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Note

Euler–Mahonian distributions of type B_n

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A B S T R A C T

Adin, Brenti, and Roichman introduced the pairs of statistics (ndes, nmaj) and (fdes, fmaj). They showed that these pairs are equidistributed over the hyperoctahedral group B_n, and can be considered "Euler–Mahonian" in the sense that they generalize the Carlitz identity. Further, they asked whether there exists a bijective proof of the equidistribution of their statistics. We give such a bijection, along with a new proof of the generalized Carlitz identity.

1. Introduction

Statistics such as inversion number, major index, and descent number have been well studied for the symmetric group S_n. MacMahon [7] showed that inversion number and major index are equidistributed and Carlitz [4] described the joint distribution of descent and major index – the "Euler–Mahonian" distribution – with the following theorem.

Theorem 1 ([4]). *For positive integers n,*

$$\sum_{r\geq 0}[r+1]_q^n t^r = \frac{\sum_{u\in S_n} t^{\mathrm{des}(u)} q^{\mathrm{maj}(u)}}{\prod_{i=0}^{n}(1-tq^i)}, \tag{1}$$

where $[r+1]_q = 1+q+\cdots+q^r$.

The Coxeter group generalization of inversion number is *length*. (An element w in a Coxeter group has length k if $w = s_1\cdots s_k$ is a minimal expression for w as a product of simple reflections.) Adin and Roichman [3] generalized MacMahon's result to the hyperoctahedral group B_n in demonstrating that a new statistic, the *flag major index*, is equidistributed with length. Adin et al. [1], introduced new statistics on B_n, the *negative descent number*, the *negative major index*, and the *flag descent number*. They proved that the pairs of statistics (fdes, fmaj) and (ndes, nmaj) are equidistributed over B_n. Moreover, they showed that these bivariate distributions are "type B_n Euler–Mahonian" in the sense that they generalize Theorem 1 as follows. (Chow and Gessel provide an alternative generalization in [5].)

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Theorem 2. *For positive integers n,*

$$\sum_{r\geq 0}[r+1]_q^n t^r = \frac{\sum_{w\in B_n} t^{\mathrm{ndes}(w)}q^{\mathrm{nmaj}(w)}}{(1-t)\prod_{i=1}^{n}(1-t^2q^{2i})} \quad [1, \text{Theorem 3.2}], \tag{2}$$

$$= \frac{\sum_{w\in B_n} t^{\mathrm{fdes}(w)}q^{\mathrm{fmaj}(w)}}{(1-t)\prod_{i=1}^{n}(1-t^2q^{2i})} \quad [1, \text{Theorem 4.2}]. \tag{3}$$

One of the drawbacks of [1] is that while their proof of (2) is quite brief and elementary, their proof of (3) is rather indirect and tedious. Further, they remark (see the discussion after [1, Corollary 4.5]) that "it would be interesting to have a direct combinatorial (i.e., bijective) proof" that these type B_n Euler–Mahonian statistics are equidistributed. The purpose of this note is to provide such a proof (Theorem 6). Along the way we give new combinatorial proofs of (2) and (3) (Theorems 3 and 4, respectively).

2. Overview

For any word $w = w_1 \cdots w_n$ whose letters are totally ordered, let $\mathrm{Des}(w) := \{i : w_i > w_{i+1}\}$, and let $\mathrm{des}(w) := |\mathrm{Des}(w)|$, called the *descent set* and *descent number*, respectively. We define the *major index* of w as

$$\mathrm{maj}(w) := \sum_{i\in \mathrm{Des}(w)} i.$$

Let S_n denote the set of all permutations of the set $[n] := \{1, 2, \ldots, n\}$. The *refined Eulerian polynomial* is the generating function for the joint distribution of (des, maj) over S_n:

$$S_n(t, q) := \sum_{u\in S_n} t^{\mathrm{des}(u)}q^{\mathrm{maj}(u)}.$$

This function is the numerator of the right-hand side of (1). For $q = 1$, we have $S_n(t) = \sum_{u\in S_n} t^{\mathrm{des}(u)}$, the classical Eulerian polynomial shifted by t^{-1}.

The hyperoctahedral group, B_n, has as its elements all *signed permutations* of $[n]$. For our purposes, these are all words $w = w_1 \cdots w_n$ on the alphabet

$$[n] \cup [\bar{n}] = \{\bar{1}, 1, \bar{2}, 2, \ldots, \bar{n}, n\}$$

such that $|w| = |w_1| \cdots |w_n|$ is a permutation in S_n. (We write $\bar{i}$ for $-i$.) The precise way in which the flag and negative statistics are defined for w in B_n depends on how we choose to put a total order on $[n] \cup [\bar{n}]$. It will be most convenient to have the lexicographic order:

$$\bar{1} < \bar{2} < \cdots < \bar{n} < 1 < 2 < \cdots < n.$$

We remark that [3] uses this order, whereas [1] uses the usual integer ordering.

Our approach is to provide two very explicit combinatorial proofs that the pairs of statistics in question are distributed over B_n as

$$S_n(t, q) \cdot \prod_{i=1}^{n}(1 + tq^i).$$

For $q = 1$ this is the intriguing formula $(1 + t)^n S_n(t)$. (The first author in fact has a different combinatorial proof that flag descents are distributed in this fashion; see [6].)

In fact, we will demonstrate a finer result, namely, that for each pair of statistics we have a way of assigning 2^n signed permutations to each unsigned permutation $u \in S_n$ such that the resulting collection has distribution

$$t^{\mathrm{des}(u)}q^{\mathrm{maj}(u)}\prod_{i=1}^{n}(1 + tq^i).$$

The signed permutations in this collection are thus identified with a pair (u, J), $J \subset [n]$, with "weight" $t^{\mathrm{des}(u)+|J|}q^{\mathrm{maj}(u)+\sum_{j\in J} j}$. This weight-preserving bijection can then be composed to obtain the desired bijection. If w corresponds to (u, J) with respect to (ndes, nmaj) and v corresponds to (u, J) with respect to (fdes, fmaj), then we identify w and v with one another:

$$\begin{tikzcd} w \arrow[rr, dashed, leftrightarrow] \arrow[dr, leftrightarrow, "\text{neg}"'] & & v \\ & (u, J) \arrow[ur, leftrightarrow, "\text{flag}"'] & \end{tikzcd}$$

This idea is formalized with Theorem 6.

The paper is organized as follows. Section 3 presents the negative statistics, Section 4 discusses the flag statistics, and Section 5 exhibits the desired bijection.

3. The negative statistics

As in [1], define the *negative descent number* of $w \in B_n$ as

$$\mathrm{ndes}(w) := \mathrm{des}(w) + |\{i : w_i < 0\}|,$$

and the *negative major index* as

$$\mathrm{nmaj}(w) := \mathrm{maj}(w) + \sum_{w_i<0} |w_i|.$$

For example if $w = \bar{5}7612\bar{4}\bar{3}$, $\mathrm{ndes}(w) = 4 + 3 = 7$, and $\mathrm{nmaj}(w) = 16 + 12 = 28$. Let

$$B_n^{(\mathrm{neg})}(t, q) := \sum_{w\in B_n} t^{\mathrm{ndes}(w)} q^{\mathrm{nmaj}(w)},$$

be the generating function for this pair of statistics.

With any $u = u_1 \cdots u_n$ in S_n we can associate 2^n signed permutations in the following manner. Define the *standardization* of a signed permutation $w \in B_n$, $\mathrm{st}(w)$, to be the unsigned permutation in S_n that is obtained by replacing the smallest letter of w with 1, the next smallest with 2, and so on. For example, $\mathrm{st}(\bar{5}7612\bar{4}\bar{3}) = 3764521$.

Now given any u in S_n, let

$$B(u) = \{w \in B_n : \mathrm{st}(w) = u\}.$$

We have

$$B_n = \bigcup_{u\in S_n} B(u) \quad \text{(disjoint union)}.$$

Remark. This partition of B_n is employed by Adin et al. (see Eq. (5) of [1]), though in different language. It plays a role in their proof of their Theorem 3.2 (Eq. (2)) and a refined version of the same result from a different work [2, Theorem 6.7]. This partition will play a role in our proof as well, though the proofs are fundamentally distinct.

It is easy to see that every element of $B(u)$ is uniquely determined by its set of negative letters. Let u_J denote the member of $B(u)$ that is a permutation of the set

$$\{i : i \in [n] \setminus J\} \cup \{\bar{j} : j \in J\}.$$

For example, if $u = 3142$, then $u_{\{1,3\}} = 2\bar{1}4\bar{3}$ since $\bar{1}$ and $\bar{3}$ are the negative letters and $\mathrm{st}(2\bar{1}4\bar{3}) = 3142$.

The following result and Theorem 1 imply Eq. (2) of Theorem 2.

Theorem 3. *We have*

$$B_n^{(\mathrm{neg})}(t, q) = S_n(t, q) \cdot \prod_{i=1}^{n} (1 + tq^i).$$

In particular, for any $u \in S_n$, we have

$$t^{\mathrm{des}(u)} q^{\mathrm{maj}(u)} \prod_{i=1}^{n} (1 + tq^i) = \sum_{w\in B(u)} t^{\mathrm{ndes}(w)} q^{\mathrm{nmaj}(w)}.$$

Proof. Given $u \in S_n$ and any $J \subset [n]$, let $w = u_J$. We first show that

$$t^{\mathrm{ndes}(w)} q^{\mathrm{nmaj}(w)} = t^{\mathrm{des}(u)} t^{\mathrm{maj}(u)} \prod_{j\in J} tq^j. \tag{4}$$

Since $\mathrm{st}(w) = u$, we have $\mathrm{Des}(w) = \mathrm{Des}(u)$. In particular, $\mathrm{des}(w) = \mathrm{des}(u)$ and $\mathrm{maj}(w) = \mathrm{maj}(u)$. By the definition of w, we know that $\{w_i : w_i < 0\} = J$. Comparing with the definition of the negative statistics, we have (4).

Summing now over all $J \subset [n]$, we get

$$\begin{aligned} \sum_{w\in B(u)} t^{\mathrm{ndes}(w)} q^{\mathrm{nmaj}(w)} &= t^{\mathrm{des}(u)} t^{\mathrm{maj}(u)} \sum_{J\subset[n]} \prod_{j\in J} tq^j \\ &= t^{\mathrm{des}(u)} t^{\mathrm{maj}(u)} \prod_{i=1}^{n} (1 + tq^i). \end{aligned}$$

This completes the proof. □

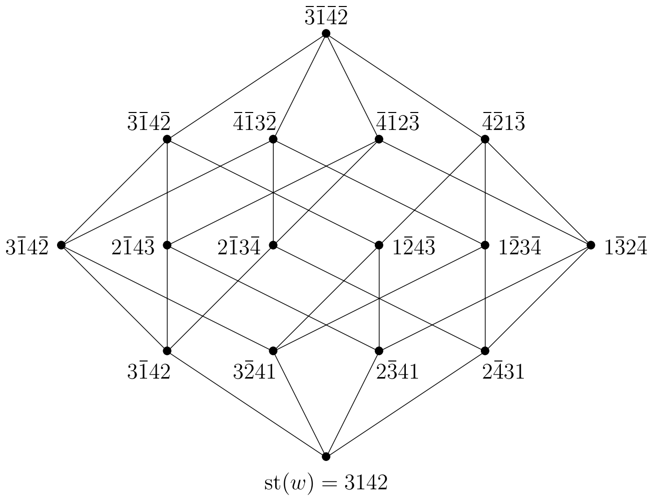

Fig. 1. $B(u)$ for $u = 3142$.

It is informative to observe that Theorem 3 implies the set $B(u)$ has, up to a shift, a weight-preserving bijection with a Boolean algebra given by

$$J \longleftrightarrow u_J.$$

That is, if $w \in B(u)$ corresponds to $J = \{j_1, \ldots, j_k\}$, then $\mathrm{ndes}(w) = \mathrm{des}(u) + k$ and $\mathrm{nmaj}(w) = \mathrm{maj}(u) + (j_1 + \cdots + j_k)$. Fig. 1 provides an illustration for $u = 3142$.

4. The flag statistics

From [1], our definitions for the flag statistics are as follows. For $w \in B_n$,

$$\mathrm{fdes}(w) := \begin{cases} 2\mathrm{des}(w) + 1 & \text{if } w_1 < 0 \\ 2\mathrm{des}(w) & \text{if } w_1 > 0, \end{cases}$$

and

$$\mathrm{fmaj}(w) := 2\mathrm{maj}(w) + |\{i : w_i < 0\}|.$$

So, for example, if $w = \bar{5}7612\bar{4}\bar{3}$, then $\mathrm{Des}(w) = \{2, 3, 5, 6\}$, $\mathrm{fdes}(w) = 2 \cdot 4 + 1 = 9$, and $\mathrm{fmaj}(w) = 2 \cdot 16 + 3 = 35$. Let $\mathrm{wt}(w) = t^{\mathrm{fdes}(w)} q^{\mathrm{fmaj}(w)}$ be the *weight* of w. The generating function is

$$B_n^{(\mathrm{flag})}(t, q) = \sum_{w \in B_n} \mathrm{wt}(w) = \sum_{w \in B_n} t^{\mathrm{fdes}(w)} q^{\mathrm{fmaj}(w)}.$$

Before proceeding, it will be helpful to provide another partition of B_n. To any $u = u_1 \cdots u_n$ in S_n, we can assign 2^n signed permutations by independently assigning minus signs to the letters of u in all possible ways. That is, let

$$B'(u) = \{w \in B_n : |w| = u\}.$$

We have

$$B_n = \bigcup_{u \in S_n} B'(u) \quad \text{(disjoint union)}.$$

Just as with Theorem 3, we have the following, which, with Theorem 1, completes the proof of Theorem 2.

Theorem 4 (*[1, Theorem 4.4]*). *We have*

$$B_n^{(\mathrm{flag})}(t, q) = S_n(t, q) \cdot \prod_{i=1}^{n} (1 + tq^i).$$

Corollary 4.1 (*[1, Corollary 4.5]*). *The statistics* (fdes, fmaj) *and* (ndes, nmaj) *are equidistributed, i.e.,*

$$B_n^{(\mathrm{flag})}(t, q) = B_n^{(\mathrm{neg})}(t, q).$$

Theorem 4 will follow from the following theorem, analogous to Theorem 3. With the combinatorial proof of Theorem 5 in hand we will be able to provide the desired bijective proof of Corollary 4.1.

Theorem 5. *For any u in S_n, we have*

$$t^{\mathrm{des}(u)}q^{\mathrm{maj}(u)}\prod_{i=1}^{n}(1+tq^i)=\sum_{w\in B'(u)}t^{\mathrm{fdes}(w)}q^{\mathrm{fmaj}(w)}.$$

Theorem 5 will follow from Lemma 5.2. First, we need one more idea. Let Δ_i be the operator that negates the first i letters of a signed permutation, i.e.,

$$\Delta_i w=\overline{w_1\cdots w_i}w_{i+1}\cdots w_n.$$

Lemma 5.1. *Fix $u\in S_n$ and let $w\in B'(u)$ be such that all the letters $w_1,\dots,w_{j+1}$ have the same sign. Then*

$$\mathrm{wt}(\Delta_j w)=\begin{cases}\mathrm{wt}(w)/tq^j & \text{if } j\in\mathrm{Des}(u)\\ \mathrm{wt}(w)\cdot tq^j & \text{if } j\notin\mathrm{Des}(u).\end{cases}$$

Proof. Suppose $w_1,w_2,\dots,w_{j+1}$ are positive. If $j\notin\mathrm{Des}(u)$ then $j\notin\mathrm{Des}(\Delta_j w)$ and $\Delta_j w$ has j new negative letters including its first letter. Hence $\mathrm{wt}(\Delta_j w)=\mathrm{wt}(w)\cdot tq^j$. If we have $j\in\mathrm{Des}(u)$, then again $\Delta_j w$ has j new negative letters, including w_1, for a total contribution of tq^j. However, since $j\notin\mathrm{Des}(\Delta_j w)$, we also lose a factor of t^2q^{2j}. Thus $\mathrm{wt}(\Delta_j w)=\mathrm{wt}(w)\cdot tq^j/t^2q^{2j}=\mathrm{wt}(w)\cdot t^{-1}q^{-j}$.

If we suppose $w_1,w_2,\dots,w_{j+1}$ are negative the situation is similar. If $j\in\mathrm{Des}(u)$ then $\Delta_j w$ has j fewer negative numbers, and w_1 has changed from negative to positive; thus $\mathrm{wt}(\Delta_j w)=\mathrm{wt}(w)\cdot t^{-1}q^{-j}$. If we have $j\notin\mathrm{Des}(u)$, then as $j\in\mathrm{Des}(\Delta_j w)$ we gain t^2q^{2j}. However, $\Delta_j w$ has j fewer negative numbers and w_1 has gone from negative to positive, yielding $\mathrm{wt}(\Delta_j w)=\mathrm{wt}(w)\cdot t^2q^{2j}/tq^j=tq^j$.

This completes the proof. □

Given any signed permutation w we have $\Delta_i^2 w=w$ and, if $i<j$,

$$\Delta_i\Delta_j w=\Delta_j\Delta_i w=w_1\cdots w_i\overline{w_{i+1}\cdots w_j}w_{j+1}\cdots w_n.$$

In particular the composite operator $\Delta_i\Delta_{i+1}$ simply negates letter $i+1$. Notice also that the Δ_i commute, so for a subset $J=\{j_1<\cdots<j_k\}$ of $[n]$, there is no ambiguity in defining the composite operator $\Delta_J=\Delta_{j_1}\cdots\Delta_{j_k}$. Moreover, if $J,K\subset[n]$, then $\Delta_J\Delta_K=\Delta_{J\Delta K}$, where $J\Delta K=(J\cup K)\setminus(J\cap K)$ is the symmetric difference of J and K.

Lemma 5.2. *For any $u\in S_n$, we have*

$$\mathrm{wt}(\Delta_J u)=t^{\mathrm{des}(u)}q^{\mathrm{maj}(u)}\prod_{j\in J\Delta\mathrm{Des}(u)}tq^j.$$

Proof. First notice that $\mathrm{fdes}(u)=2\mathrm{des}(u)$ and $\mathrm{fmaj}(u)=2\mathrm{maj}(u)$. Since the Δ_j commute, we can apply Δ_{j_k} first, $\Delta_{j_{k-1}}$ second, and so on. This allows us to apply Lemma 5.1 repeatedly, giving us

$$\begin{aligned}\mathrm{wt}(\Delta_J u)&=t^{2\mathrm{des}(u)}q^{2\mathrm{maj}(u)}\frac{\prod_{j\in J\setminus\mathrm{Des}(u)}tq^j}{\prod_{i\in J\cap\mathrm{Des}(u)}tq^i}\\&=t^{\mathrm{des}(u)}q^{\mathrm{maj}(u)}\prod_{j\in J\Delta\mathrm{Des}(u)}tq^j,\end{aligned}$$

as desired. □

Now we can prove Theorem 5.

Proof of Theorem 5. We first claim that

$$B'(u)=\{\Delta_J u : J\subset[n]\}.$$

Because the two sets have the same cardinality, it suffices to show $\Delta_J u\in B'(u)$ for any J. This is clear since $|\Delta_J u|=u$.

It now follows from Lemma 5.2 that

$$\begin{aligned}\sum_{w\in B'(u)}t^{\mathrm{fdes}(w)}q^{\mathrm{fmaj}(w)}&=\sum_{J\subset[n]}\mathrm{wt}(\Delta_J u)\\&=t^{\mathrm{des}(u)}q^{\mathrm{maj}(u)}\sum_{J\subset[n]}\prod_{j\in J\Delta\mathrm{Des}(u)}tq^j\\&=t^{\mathrm{des}(u)}q^{\mathrm{maj}(u)}\prod_{i=1}^{n}(1+tq^i),\end{aligned}$$

proving Theorem 5. □

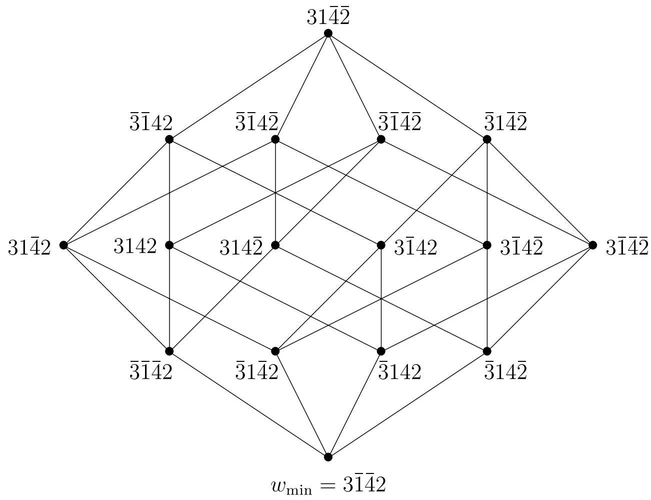

Fig. 2. $B'(u)$ for $u = 3142$.

5. The bijection

From Lemma 5.2, we notice in particular that there is a unique signed permutation with weight $t^{\mathrm{des}(u)}q^{\mathrm{maj}(u)}$, obtained by taking $J = \mathrm{Des}(u)$. Define

$$w_{\min}(u) = \Delta_{\mathrm{Des}(u)}u.$$

Since the Δ_i commute and generate all possible sign changes, we can identify permutations in $B'(u)$ with subsets of $[n]$ by the correspondence

$$J \longleftrightarrow \Delta_J w_{\min}(u) = \Delta_{J\triangle \mathrm{Des}(u)}u.$$

Lemma 5.2 can now be interpreted to show that the Boolean algebra generated by subsets of $[n]$ respects the statistics (fdes, fmaj) on $B'(u)$. Specifically, if $w \in B'(u)$ corresponds to $J = \{j_1, \dots, j_k\}$, then $\mathrm{fdes}(w) = \mathrm{des}(u) + k$ and $\mathrm{fmaj}(w) = \mathrm{maj}(u) + (j_1 + \cdots + j_k)$.

See Fig. 2 for an example with $u = 3142$.

Implicit here is the bijection that Adin et al. wanted.

Theorem 6. *The bijection*

$$u_J \longleftrightarrow \Delta_{J\triangle \mathrm{Des}(u)}u$$

is weight-preserving in that

$$\mathrm{ndes}(u_J) = \mathrm{fdes}(\Delta_{J\triangle \mathrm{Des}(u)}u)$$
$$\mathrm{nmaj}(u_J) = \mathrm{fmaj}(\Delta_{J\triangle \mathrm{Des}(u)}u).$$

Example. Let $w = \bar{2}1\bar{3}\bar{4}$ with $\mathrm{ndes}(w) = 4$ and $\mathrm{nmaj}(w) = 11$. We first find $u = \mathrm{st}(w) = 1423$. Since $J = \{2, 3, 4\}$ is the set of negative letters in w and $\mathrm{Des}(u) = \{2\}$, we take $v = \Delta_3\Delta_4(1423) = 142\bar{3}$, with the desired flag descent and flag major numbers, $\mathrm{fdes}(v) = 4$ and $\mathrm{fmaj}(v) = 11$.

In the other direction, let $v = 43\bar{1}\bar{2}$ with $\mathrm{fdes}(v) = 4$ and $\mathrm{fmaj}(v) = 8$. First we see that $u = |v| = 4312$ and $v = \Delta_2\Delta_4 u$. Since $\mathrm{Des}(u) = \{1, 2\}$ and $J\triangle\mathrm{Des}(u) = \{2, 4\}$, we get $J = \{1, 4\}$ and $u_J = 32\bar{1}\bar{4}$ with $\mathrm{ndes}(u_J) = 4$ and $\mathrm{nmaj}(u_J) = 8$, as desired.

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